Journal of Algebra, Number Theory: Advances and Applications
Volume 3, Number 1, 2010, Pages 1-4

ON FACTORIZATION OF FINITE GROUPS

A. MAHMIANI

Islamic Azad University Aliabad Katool Branch
Aliabad Katool
Iran
e-mail: mahmiani_a@yahoo.com

Abstract

A group G is called non-factorizable, if $G \neq 1$ and for any subgroup A of G, there is no subgroup B of G such that $G = AB$. In this paper, we will prove that, if G is a non-factorizable group with Frattini subgroup of order p^2, where p is a prime number, then G is a quasi-simple group.

1. Introduction

A group G is called *non-factorizable*, if $G \neq 1$ and for any subgroup A of G, there is no subgroup B of G such that $G = AB$. If $G \neq 1$ and for all proper subgroups A of G, a subgroup B of G does not exist such that $G = AB$. Then G is called a *non-factorizable group*. If there is a subgroup B of G such that $G = AB$, then B is called a *supplement* of A in G, and if in addition, $A \cap B = 1$, then B is called a *complement* of A in G.

In [1] page 13, the question of finding all the factorizable group is raised. As a matter of fact, every group is not factorizable, for example, by [4], the Mathien group of degree 22, M_{22}, is a non-factorizable group.

2010 Mathematics Subject Classification: 20D40.

Keywords and phrases: factorization, linear fractional group, symmetric group.

Received December 29, 2009

One aspect of finding factorizable groups is the problem of involvement of the alternating group in a proper factorization of a finite group G.

To quote a few papers, we mention that in [5], finite groups $G = AB$ with $A \cong B \cong A_5$ are found; in [7], finite factorizable groups with one factor a non-abelian simple group and the other factor isomorphic to the alternating group on 5 letters are found. In [2], all factorizable groups with one factor isomorphic to the alternating group on 6 letters, and the other factor isomorphic to the symmetric group on $n \geq 5$ letters are found. But, in this paper, we are interested in non-factorizable groups. In [3], the following result concerning non-factorizable groups is proved.

Result 1. Let G be a non-factorizable group. Then, G is of the following three types of groups:

(i) a cyclic p-group,

(ii) a non-abelian non-factorizable simple group,

(iii) a perfect group with $\phi(G) \neq 1$ and $\frac{G}{\phi(G)}$, a non-abelian non-factorizable simple group.

In the above result, $\phi(G)$ denotes the Frattini subgroup of G, which by definition is the intersection of all the maximal subgroups of G. And G is called a *perfect group*, if it coincides with its commutator subgroup, i.e., $G' = G$. Refering to Result 1, we observe that groups in items (i) and (ii) are known. Groups of type (ii) are listed in [4]. But, groups in item (iii) need investigation. Therefore, groups satisfying condition (iii) of the theorem are called *type* III *groups*. In [3], it is proved that, if G is finite group of type III with $|\phi(G)| = p$, p is a prime number, then $\phi(G) = Z(G)$ and G is quasi-simple. We recall that a group G is called *semi-simple*, if G is perfect and $\frac{G}{Z(G)}$ is simple. In this paper, the following theorem is proved:

Theorem 1. *Let p be a prime. If G is a type* III *group with $|\phi(G)| = p^2$, then G is a quasi-simple group.*

Proof. Since $\phi(G)$ is abelian, we have $\phi(G) \trianglelefteq C_G(\phi(G)) =\triangleleft\, G$. Therefore, $\frac{C_G(\phi(G))}{\phi(G)}$ is a normal subgroup of $\frac{G}{\phi(G)}$, hence $C_G(\phi(G)) = \phi(G)$ or G.

Case 1. $C_G(\phi(G)) = \phi(G)$.

In this case, $\frac{G}{\phi(G)}$ is isomorphic to a subgroup of $Aut(\phi(G))$. As $|\phi(G)| = p^2$, $\phi(G)$ is either cyclic or an elementary abelian p-group.

If $\phi(G) \cong Z_{p^2}$, a cyclic group of order p^2, then $Aut(\phi(G)) \cong Z_{p(p-1)}$, when p is odd, and $Aut(\phi(G)) \cong Z_2$, in the case of $p = 2$. But, it is easy to see that in both cases $\frac{G}{\phi(G)}$ can not be a simple group.

If $\phi(G) \cong Z_p \times Z_p$, then $\frac{G}{\phi(G)}$ is isomorphic to a non-abelian simple subgroup of $GL_2(p)$. By [6] page 404, the only possibility for $\frac{G}{\phi(G)}$ is $SL_2(p)$. Of course, in this case, p must be odd. The group G is isomorphic to the special affine group $G \cong ASL_2(p)$, which act on the vectors of the underlying vector space.

$\phi(G) \cong V_2(p)$ according to the rule:

$$T_{A,b}(\nu) = A\nu + b,$$

where $A \in SL_2(p)$, $b \in V_2(p)$, and $T_{A,b}$ is an element of $ASL_2(p)$. The group $ASL_2(p)$ acts 2-transitivity on the set of p^2 vectors of $V_2(p)$, and it is easy to verify that this action is 2-transitive. Therefore, the action of G on $V_2(p)$ is primitive and the stabilizer of each vector is a maximal subgroup of G. Therefore, $\phi(G)$ is contained in the kernel of this action, which is the trivial group. Hence $\phi(G) = 1$, a contradiction.

Case 2. $C_G(\phi(G)) = G$.

In this case, we deduce that $\phi(G) \leq Z(G)$. Since $\frac{Z(G)}{\phi(G)} \trianglelefteq \frac{G}{\phi(G)}$ and $\frac{G}{\phi(G)}$ is a non-abelian simple group, we deduce that $\phi(G) = Z(G)$. But, this will imply that $\frac{G}{Z(G)}$ is a quasi-simple group, and the theorem is proved. □

References

[1] B. Amberg, S. Franciosi and F. Degiovann, Products of Groups, Oxford University Press, 1992.

[2] M. R. Darafcheh and G. R. Rezaeezadeh, Factorization of groups involving symmetric and alternating groups, Int. J. Math. Math. Sci. 27(3) (2001), 161-167.

[3] J. Kirtland, Finding proper factorizations in finite groups, Algebra and Discrete Mathematics 2 (2009), 45-59.

[4] M. W. Liebeck, C. E. Praeger and J. Saxl, The maximal factorizations of the finite simple groups and their automorphism groups, Mem. AMS 86(432) (1990).

[5] W. R. Scott, Products of A_5 and a finite simple group, J. Alg. 37 (1975), 165-171.

[6] M. Suzuki, Group Theory I, Springer-Verlag, 1982.

[7] G. L. Walls, Products of simple groups and symmetric groups, Arch. Math. 58 (1992), 313-321.

■